Polyhedral spaces, with a twist

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Contents

Brief historical account on polyhedral spaces

A Banach space X is *polyhedral* if the unit ball of every finite-dimensional subspace of X is the convex hull of finitely many points.

(1) *V. Klee (1959 – 1960)*

- c_0 is polyhedral.
- Is there an infinite-dimensional reflexive polyhedral space?

(2) *J. Lindenstrauss (1966)*

- No infinite-dimensional *dual* space can be polyhedral.

(3) *V. P. Fonf.* (and collaborators)

- Every polyhedral space is c_0-*saturated* (1979 – 1981).
- ...and much more.

A relaxation

Polyhedrality is an *isometric* notion, and this can be sometimes inconvenient.

- For instance, in a diagram

$$0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$$

we do not necessarily know "the" norm of Z.

Definition

A Banach space is said to be *isomorphically polyhedral* if it admits a polyhedral renorming.

Main question:

Is "to be isomorphically polyhedral" a three-space property?

Examples of (isomorphically) polyhedral spaces

(1) c_0 is polyhedral, but c is not.

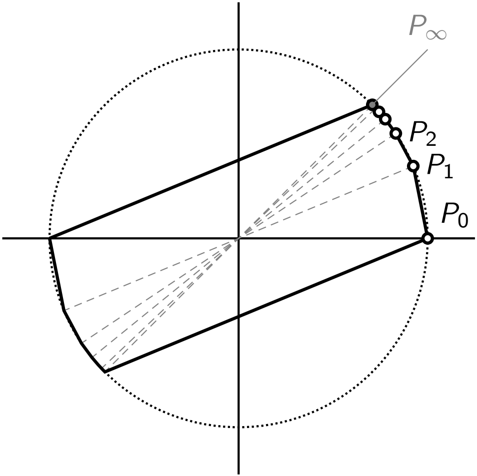

Figure: extracted from A. J. Guirao, V. Montesinos and V. Zizler.

Examples of (isomorphically) polyhedral spaces

(1) c_0 is polyhedral, but c is not.
In fact, no $C(K)$-space can be polyhedral in the usual norm.

(2) If K is *σ-discrete*, then $C(K)$ is isomorphically polyhedral. This includes
- $c_0(I)$ for any set I.
- $C(K)$ for K scattered of countable height.

(3) If a $C(K)$-space admits a polyhedral renorming, then K must be scattered (equivalently, $C(K)$ Asplund).

(4) The converse is (consistently) not true: Kunen's compact space.

Question 1

Characterize compact spaces K such that $C(K)$ is isomorphically polyhedral.

Polyhedrality vs. $\mathcal{L}_\infty$-spaces

(1) $C[0,1]$ is an $\mathcal{L}_\infty$-space which is not isomorphically polyhedral.

(2) $c_0(\ell_2^n)$ or the Schreier space are isomorphically polyhedral spaces which are not $\mathcal{L}_\infty$-spaces.

(3) However, preduals of ℓ_1 are isomorphically polyhedral.

(4) But this is no longer (consistently) true for preduals of $\ell_1(\Gamma)$.

It is important to set boundaries

A (James) *boundary* for a Banach space $(X, \|\cdot\|)$ is a subset $B \subseteq S_{X^*}$ such that for every $x \in X$ there is $x^* \in B$ with $x^*(x) = \|x\|$.

A boundary B for a Banach space X has *property* $(\star)$ if every weak*-cluster point x^* of B satisfies $x^*(x) < 1$ for every $x \in B_X$.

Proposition (Fonf, Smith, Pallarés, Troyanski)

Every Banach space admitting a boundary with property $(\star)$ is isomorphically polyhedral.

It is important to set boundaries

Actually, a *separable* Banach space is isomorphically polyhedral if and only if it has a boundary with property $(\star)$.

Question 2

Does the existence of a boundary with property $(\star)$ characterizes isomorphically polyhedral Banach spaces in general?

A bit on twisted sums

The diagram

$$0 \longrightarrow Y \xrightarrow{j} Z \xrightarrow{q} X \longrightarrow 0$$

means:

- j is an into isomorphism.
- $j(Y) = \ker q$.
- q is a quotient map.

We say an exact sequence *splits* whenever $j(Y)$ is complemented in Z, or (equivalently) if q admits a linear and bounded selection.

Twisted sums of $c_0(I)$-spaces

(1) $C(K_{\mathcal{A}})$-spaces.

$$0 \longrightarrow c_0 \longrightarrow C(K_{\mathcal{A}}) \longrightarrow c_0(\mathcal{A}) \longrightarrow 0$$

What is $K_{\mathcal{A}}$?

- Let $\mathcal{A} = \{A_i\}_{i \in I} \subseteq \mathcal{P}(\mathbb{N})$ with the property that $A_i \cap A_j$ is finite for every $i, j \ \in I$.
- Define a topology on
$$\Psi_{\mathcal{A}} = \mathbb{N} \cup \{p_A : A \in \mathcal{A}\}$$
by (essentially):
$$(n)_{n \in A} \to p_A \ \text{ for every } A \in \mathcal{A}$$
- $K_{\mathcal{A}}$ is the one-point compactification of $\Psi_{\mathcal{A}}$.

Proposition

$C(K_{\mathcal{A}})$ is isomorphically polyhedral.

Twisted sums of $c_0(I)$-spaces

(1) $C(K_{\mathcal{A}})$-spaces

$$0 \longrightarrow c_0 \longrightarrow C(K_{\mathcal{A}}) \longrightarrow c_0(\mathcal{A}) \longrightarrow 0$$

(2) More exotic Banach spaces:

$$0 \longrightarrow c_0 \longrightarrow PS_2 \longrightarrow c_0(\mathfrak{c}) \longrightarrow 0$$

- PS_2 produces a counterexample to the *complemented subspace problem* for $C(K)$-spaces...
- ...and also for Banach lattices.

Proposition

PS_2 is isomorphically polyhedral.

The main theorem

Question: are twisted sums of $c_0(I)$-spaces isomorphically polyhedral?

Yes! And we can do a little better:

Theorem (with J. M. F. Castillo)

Assume X admits a boundary with property $(\star)$. Then every twisted sum Z of $c_0(I)$ and X admits a boundary with property $(\star)$. In particular, Z is isomorphically polyhedral.

The proof stems from a *representation theorem* for twisted sums of $c_0(I)$ and any Banach space X.

How do we obtain twisted sums of $C(K)$-spaces?

One basic idea: take K and L compact spaces such that

- $K \hookrightarrow L$.
- $L \setminus K$ is discrete of size κ.

And form the exact sequence

$$0 \longrightarrow c_0(\kappa) \longrightarrow C(L) \longrightarrow C(K) \longrightarrow 0$$

In such a case, we say L is a *discrete extension* of K. This is often written as $L = K \cup \kappa$.

A representation theorem

It is no hope that every twisted sum of c_0 and $C(K)$ comes from a discrete extension of K.

Theorem 1 (with J. M. F. Castillo)

For every twisted sum Z of $c_0(\kappa)$ and X, there is a discrete extension L of (B_{X^*}, w^*) such that

$$\begin{tikzcd}
0 \arrow[r] & c_0(\kappa) \arrow[r] & C(L) \arrow[r] & C(B_{X^*}) \arrow[r] & 0 \\
0 \arrow[r] & c_0(\kappa) \arrow[r] \arrow[u, equal] & Z \arrow[r] \arrow[u, hook] & X \arrow[r] \arrow[u, hook, "\phi"] & 0
\end{tikzcd}$$

where $\phi : X \hookrightarrow C(B_{X^*})$ is the natural embedding.

A bit into the proof

- Consider

$$0 \longrightarrow c_0(\kappa) \xrightarrow{j} Z \xrightarrow{q} X \longrightarrow 0$$

- The dual exact sequence must split:

$$0 \longrightarrow X^* \xrightarrow{q^*} Z^* \xrightarrow{j^*} \ell_1(\kappa) \longrightarrow 0$$

- Hence there is a bounded set $(z_k^*)_{k\in\kappa}$ satisfying that $j^*(z_k^*) = e_k^*$ for all $k \in \kappa$.
- No z_k^* lies in $q^*(X^*)$, but w*-cluster points of $(z_k^*)_{k\in\kappa}$ do lie in (a multiple of) $q^*(B_{X^*})$.
- Let $L = q^*(B_{X^*}) \ \cup \{z_k^* : k \in \kappa\}$, and check.

Back to the main theorem

We are ready to prove:

Theorem (with J. M. F. Castillo)

Assume X admits a boundary with property $(\star)$. Then every twisted sum Z of $c_0(I)$ and X admits a boundary with property $(\star)$. In particular, Z is isomorphically polyhedral.

Sketch of the proof

- Consider a twisted sum of c_0 and X, which now involves

$$\begin{tikzcd} 0 \arrow[r] & c_0(\kappa) \arrow[r] & C(B_{X^*} \cup \kappa) \arrow[r] & C(B_{X^*}) \arrow[r] & 0 \\ 0 \arrow[r] & c_0(\kappa) \arrow[r] \arrow[u, equal] & Z \arrow[r] \arrow[u, hook, "u"] & X \arrow[r] \arrow[u, hook, "\phi"] & 0 \end{tikzcd}$$

In particular, $\|z\| = \sup_{t\in L} |u^*\delta_t(z)|$.

- If B is a boundary for X, then

$$\{\pm u^*\delta_k : k \in \kappa\} \cup \{u^*\delta_{b^*} : b^* \in B\}$$

is a boundary for Z.

1 is different from 2

- But it is unclear whether it has property $(*)$ if B does.
- So we consider

$$V = \{\pm u^* \delta_k : k \in \kappa\} \cup \{2u^* \delta_{b^*} : b^* \in B\}$$

and define a new norm on Z by

$$|||z||| = \sup_{v \in V} |v(z)|$$

- Now, if f^* is a weak*-cluster point of V, then it is of the form $\lambda \cdot \delta_{x^*}$, for a certain $x^* \in X^*$ and $\lambda \in \{1, 2\}$.
- Either way, it cannot be so that $f^*(x) = 1$ if $|||x||| \leq 1$.

A detour: the structure of twisted sums of $c_0(I)$

Consider a twisted sum

$$0 \longrightarrow c_0(\kappa) \xrightarrow{j} Z \xrightarrow{q} X \longrightarrow 0$$

Then:

(1) Z can be renormed to be an $\mathcal{L}_{\infty,1^+}$-space provided X is an $\mathcal{L}_{\infty,1^+}$-space.

(2) If $X = c_0(I)$, either Z is a subspace of $\ell_\infty(\kappa)$ or there is a copy of $c_0(\kappa^+)$ inside Z such that the restriction of q is an isomorphism.

Other candidates: twisted sums of $C(\omega^\omega)$

- Our proof relies heavily on the fact that every twisted sum of $c_0(I)$-spaces can be embedded in a "natural $C(K)$-space.
- It is unlikely that a representation theorem for twisted sums of $C(\omega^\omega)$ is true:

Example (Cabello, Castillo, Kalton, Yost)

There is a twisted sum

$$0 \longrightarrow C(\omega^\omega) \longrightarrow Z \xrightarrow{q} C(\omega^\omega) \longrightarrow 0$$

in which the quotient map q is strictly singular.

In particular:

- Z^* is isomorphic to ℓ_1.
- But Z is not isomorphic to an ℓ_1-predual.

Future perspectives

Question 3

Is every twisted sum of $C(\omega^\omega)$ and $C(\omega^\omega)$ isomorphically polyhedral?

Despite our results, we still do not know:

Question 4

If X is isomorphically polyhedral, is every twisted sum of $c_0(I)$ and X isomorphically polyhedral?

Thank you very much for your attention!