COMPACT EXACT SEQUENCES
WITH APPLICATIONS

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MOTIVATION AND OUTLINE

Let $\mathfrak{K}$ denote the ideal of compact operators. Given the following situation

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d, "T"'] & B \arrow[ld, dashed] \\
Y &
\end{tikzcd}$$

(1) Assume $T \in \mathfrak{K}$. When can T be extended? When can T be extended to a compact operator?

(2) Consider the following properties:
  - All operators $T : A \to Y$ admit extension to B.
  - All compact operators $T : A \to Y$ admit a compact extension to B.

  Does any of these imply the other?

What we will do

(i) Construct the spaces of *(compact) exact sequences*

(ii) Show how the above questions can be naturally stated in this language.

(iii) Answer them (the best we can).

SHORT EXACT SEQUENCES

- A *short exact sequence* is a diagram

$$0 \longrightarrow Y \xrightarrow{j} Z \xrightarrow{q} X \longrightarrow 0$$

in which the kernel of every arrow coincides with the image of the preceding.

- A short exact sequence is *trivial* if
  - either there is $P : Z \to Y$ with $Pj = \mathrm{Id}_Y$.
  - or there is $S : Y \to Z$ with $Sq = \mathrm{Id}_Z$.
    (both are equivalent).

- Two short exact sequences

$$0 \to Y \to Z_i \to X \to 0 \ , \quad i \in \{1, 2\},$$

are *equivalent* if there exists an operator (necessarily an isomorphism) $u : Z_1 \to Z_2$ such that

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & Z_1 \arrow[r] \arrow[d, "u"] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r] & Z_2 \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

THE SPACE OF SHORT EXACT SEQUENCES

$$\operatorname{Ext}(X,Y) = \frac{\{\text{short exact sequences} \quad 0 \to Y \to Z \to X \to 0\ \}}{\equiv}$$

- We write $\operatorname{Ext}(X,Y) = 0$ to mean that every short exact sequence of the form

$$0 \to Y \to Z \to X \to 0$$

  is trivial.
- Notation:

$$0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0 \qquad [\mathrm{z}]$$

Some classical theorems

- $\operatorname{Ext}(X,Y) = 0$ for all $Y \iff X$ is projective $\iff X = \ell_1(\Gamma)$.
- $\operatorname{Ext}(X,Y) = 0$ for all $X \iff Y$ is injective.

EXAMPLES

- A *projective presentation* of X:

$$0 \longrightarrow \ker \pi \longrightarrow \ell_1(\Gamma) \xrightarrow{\pi} X \longrightarrow 0$$

- An *injective presentation* of Y:

$$0 \longrightarrow Y \xrightarrow{\iota} \ell_\infty(\Gamma) \longrightarrow \ell_\infty(\Gamma)/Y \longrightarrow 0$$

- *The most famous* non-trivial short exact sequence:

$$0 \longrightarrow c_0 \longrightarrow \ell_\infty \longrightarrow \ell_\infty/c_0 \longrightarrow 0$$

THE PUSH-OUT SEQUENCE

Fix a short exact sequence $0 \to Y \to Z \to X \to 0$.

- Given an operator $T : Y \to Y'$,

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "j"] \arrow[d, "T"] & Z \arrow[r, "q"] & X \arrow[r] & 0 \\
 & Y' & & &
\end{tikzcd}$$

THE PUSH-OUT SEQUENCE

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$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "j"] \arrow[d, "T"] & Z \arrow[r, "q"] \arrow[d] & X \arrow[r] & 0 \\
& Y' \arrow[r] & PO & &
\end{tikzcd}$$

  where $PO = \dfrac{Z \oplus Y'}{\overline{\{(jy, -Ty) : y \in Y\}}}$.

The push-out sequence

Fix a short exact sequence $0 \to Y \xrightarrow{j} Z \xrightarrow{q} X \to 0$.

- Given an operator $T : Y \to Y'$, the *push-out sequence* is the lower row of the diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "j"] \arrow[d, "T"] & Z \arrow[r, "q"] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y' \arrow[r] & PO \arrow[r] & X \arrow[r] & 0
\end{tikzcd},$$

  where $PO = \dfrac{Z \oplus Y'}{\overline{\{(jy, -Ty) : y \in Y\}}}$.

- The push-out sequence is trivial iff $T : Y \to Y'$ admits an *extension* to Z.
- Furthermore, every diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "j"] \arrow[d, "T"] & Z \arrow[r, "q"] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

  is a push-out diagram.

THE PULL-BACK SEQUENCE

Fix a short exact sequence $0 \to Y \xrightarrow{j} Z \xrightarrow{q} X \to 0$.

- Given an operator $S : X' \to X$, the *pull-back sequence* is the lower row of the diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "j"] & Z \arrow[r, "q"] & X \arrow[r] & 0 \\
0 \arrow[r] & Y \arrow[r] \arrow[u, equal] & PB \arrow[r] \arrow[u] & X' \arrow[r] \arrow[u, "S"] & 0
\end{tikzcd}$$

  where $PB = \{(z, x') \in Z \oplus X' : qz = Sx'\}$.

- The pull-back sequence is trivial iff $S : X' \to X$ admits a *lifting* to Z.

- Furthermore, every diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] & Z \arrow[r] & X \arrow[r] & 0 \\
0 \arrow[r] & Y \arrow[r] \arrow[u, equal] & Z' \arrow[r] \arrow[u] & X' \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

  is a pull-back diagram.

“EXT” AS A SPACE OF OPERATORS

- Every short exact sequence can be obtained as a push-out of a *projective presentation of* X:

$$\begin{tikzcd}
0 \arrow[r] & \ker\pi \arrow[r] & \ell_1(\Gamma) \arrow[r, "\pi"] & X \arrow[r] \arrow[d, equal] & 0 & {[\mathrm{p}]} \\
0 \arrow[r] & Y \arrow[r, "j"] & Z \arrow[r, "q"] & X \arrow[r] & 0 &
\end{tikzcd}$$

“Ext” as a space of operators

- Every short exact sequence can be obtained as a push-out of a *projective presentation of* X:

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0 \arrow[r] & Y \arrow[r, "j"] & Z \arrow[r, "q"] & X \arrow[r] & 0 &
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“Ext” as a space of operators

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$$\begin{tikzcd}
0 \arrow[r] & \ker \pi \arrow[r] \arrow[d, "T"] & \ell_1(\Gamma) \arrow[r] \arrow[d, "\bar{T}"] & X \arrow[r] \arrow[d, equal] & 0 & {[\mathrm{p}]} \\
0 \arrow[r] & Y \arrow[r, "j"] & Z \arrow[r, "q"] & X \arrow[r] & 0 & {[T\mathrm{p}]}
\end{tikzcd}$$

- Furthermore, the sequences $[T\mathrm{p}]$ and $[T'\mathrm{p}]$ are equivalent if and only if, $T - T'$ admits an extension to $\ell_1(\Gamma)$.
- This yields a natural correspondence

$$\mathrm{Ext}(X, Y) = \frac{L(\ker \pi, Y)}{\text{extensible operators to } \ell_1(\Gamma)}.$$

"EXT" AS A SPACE OF OPERATORS, DUAL VERSION

- Also, every short exact sequence is a pull-back of an *injective presentation of Y*:

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] & \ell_\infty(\Gamma) \arrow[r] & \ell_\infty(\Gamma)/Y \arrow[r] & 0 & {[\mathtt{i}]} \\
0 \arrow[r] & Y \arrow[r] \arrow[u, equal] & Z \arrow[r] \arrow[u] & X \arrow[r] \arrow[u, "S"] & 0 & {[\mathtt{i}S]}
\end{tikzcd}$$

- Furthermore, the sequences $[\mathtt{i}S]$ and $[\mathtt{i}S']$ are equivalent if and only if, $S - S'$ admits a lifting to $\ell_\infty(\Gamma)$.
- This yields

$$\mathrm{Ext}(X, Y) = \frac{L(X, \ell_\infty(\Gamma)/Y)}{\text{liftable operators to } \ell_\infty(\Gamma)}.$$

WHY SHORT EXACT SEQUENCES?

- Given a short exact sequence $0 \to Y \xrightarrow{j} Z \xrightarrow{q} X \to 0$, the sequence

  $$\begin{tikzcd} 0 \arrow[r] & L(X,E) \arrow[r, "q^*"] & L(Z,E) \arrow[r, "j^*"] & L(Y,E) \end{tikzcd}$$

  is not usually exact on the rightmost end.
- In fact, j^* is onto precisely when E is injective.
- So we would like to continue it to make it exact:

$$\begin{tikzcd} 0 \arrow[r] & L(X,E) \arrow[r] & L(Z,E) \arrow[r] & L(Y,E) \arrow[dll, rounded corners] \\ & \operatorname{Ext}(X,E) \arrow[r] & \operatorname{Ext}(Z,E) \arrow[r] & \operatorname{Ext}(Y,E) \end{tikzcd}$$

Compact exact sequences

Recall that $\mathfrak{K}$ denotes the ideal of compact operators.

Definition

A short exact sequence is *compact* if it can be fitted in:

- a diagram (1) with $T \in \mathfrak{K}$,

$$\begin{tikzcd}
0 \arrow[r] & \ker \pi \arrow[r] \arrow[d, "T"] & \ell_1(\Gamma) \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r] & Z \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

- or in a diagram (2) with $S \in \mathfrak{K}$.

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] & \ell_\infty(\Gamma) \arrow[r] & \ell_\infty(\Gamma)/Y \arrow[r] & 0 \\
0 \arrow[r] & Y \arrow[r] \arrow[u, equal] & Z \arrow[r] \arrow[u] & X \arrow[r] \arrow[u, "S"] & 0
\end{tikzcd}$$

COMPACT EXACT SEQUENCES

This definition works because $\mathfrak{K}$ is a *balanced* operator ideal:

Proposition 1

In a diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker\pi \arrow[r] \arrow[d, "T"] & \ell_1(\Gamma) \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & Z \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "S"] & 0 \\
0 \arrow[r] & Y \arrow[r] & \ell_\infty(\Gamma) \arrow[r] & \ell_\infty(\Gamma)/Y \arrow[r] & 0
\end{tikzcd}$$

- *T can be chosen in $\mathfrak{K}$ if and only if S can be chosen in $\mathfrak{K}$.*
- *T can be chosen $\mathfrak{K}$-extensible if and only if S can be chosen $\mathfrak{K}$-liftable.*

SPACES OF COMPACT EXACT SEQUENCES

We define

$$\mathrm{Ext}_{\mathfrak{K}}(X, Y) = \frac{\mathfrak{K}(X, \ell_\infty(\Gamma)/Y)}{\mathfrak{K}\text{-liftable operators to } \ell_\infty(\Gamma)}$$

SPACES OF COMPACT EXACT SEQUENCES

We define

$$\mathrm{Ext}_{\mathfrak{K}}(X, Y) = \frac{\mathfrak{K}(X, \ell_\infty(\Gamma)/Y)}{\mathfrak{K}\text{-liftable operators to } \ell_\infty(\Gamma)}$$

$$= \frac{\mathfrak{K}(\ker \pi, Y)}{\mathfrak{K}\text{-extensible operators to } \ell_1(\Gamma)}$$

SPACES OF COMPACT EXACT SEQUENCES

We define

$$\begin{aligned}\mathrm{Ext}_{\mathfrak{K}}(X,Y) &= \frac{\mathfrak{K}(X,\ell_\infty(\Gamma)/Y)}{\mathfrak{K}\text{-liftable operators to } \ell_\infty(\Gamma)}\\ &= \frac{\mathfrak{K}(\ker\pi, Y)}{\mathfrak{K}\text{-extensible operators to } \ell_1(\Gamma)}\\ &= \frac{\{\text{compact short exact sequences} \quad 0 \to Y \to Z \to X \to 0\ \}}{\equiv_{\mathfrak{K}}}.\end{aligned}$$

- $\mathrm{Ext}_{\mathfrak{K}}(X,Y)=0$ means that every compact operator $\ker\pi \to Y$ is $\mathfrak{K}$-extensible. Equivalently, every compact operator $X \to \ell_\infty(\Gamma)/Y$ is $\mathfrak{K}$-liftable.

SPACES OF COMPACT EXACT SEQUENCES

Given $0 \to Y \xrightarrow{j} Z \xrightarrow{q} X \to 0$:

- The sequence

$$0 \to \mathfrak{K}(X,E) \xrightarrow{q^*} \mathfrak{K}(Z,E) \xrightarrow{j^*} \mathfrak{K}(Y,E)$$

  is well defined and exact.
- (Lindenstrauss) j^* is onto $\iff$ E is an $\mathcal{L}_\infty$-space.
- There is a (longer) exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{K}(X,E) \arrow[r] & \mathfrak{K}(Z,E) \arrow[r] & \mathfrak{K}(Y,E) \arrow[dll, rounded corners, to path={-- ([xshift=2ex]\tikztostart.east) |- ([yshift=-1ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
& \mathrm{Ext}_{\mathfrak{K}}(X,E) \arrow[r] & \mathrm{Ext}_{\mathfrak{K}}(Z,E) \arrow[r] & \mathrm{Ext}_{\mathfrak{K}}(Y,E)
\end{tikzcd}$$

$\operatorname{Ext}(X, Y) = 0$ vs $\operatorname{Ext}_{\mathfrak{K}}(X, Y) = 0$

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "T"'] & B \arrow[r] \arrow[dl, dashed, "\hat{T}"] & X \arrow[r] & 0 \\
& Y & & &
\end{tikzcd}$$

Theorem 1

Let X and Y be Banach spaces.

(a) *If Y has the BAP, then* $\operatorname{Ext}(X, Y) = 0$ *implies* $\operatorname{Ext}_{\mathfrak{K}}(X, Y) = 0$.

(b) *If Y is complemented in its bidual and either Y or X has the BAP, then* $\operatorname{Ext}_{\mathfrak{K}}(X, Y) = 0$ *implies* $\operatorname{Ext}(X, Y) = 0$.

The hypothesis on complementation is really necessary:

$$\operatorname{Ext}(\ell_\infty/c_0, c_0) \neq 0, \qquad \operatorname{Ext}_{\mathfrak{K}}(\ell_\infty/c_0, c_0) = 0.$$

Question 1

Is the BAP really necessary in Theorem 1?

(Compact) extensions of compact operators

$$\begin{tikzcd}
Y \arrow[r, hook] \arrow[d, "\alpha"'] \arrow[dr, "\alpha'"] & Z \arrow[d, dashed] \\
E & F
\end{tikzcd}$$

Theorem 2

If $\alpha : Y \to E$ is a compact operator that cannot be extended to Z, then there is a compact operator $\alpha' : Y \to F$ such that:

- *α' can be extended to Z.*
- *No extension of α' to Z is compact.*

Sketch of proof

- Since α cannot be extended, the push-out row is not trivial:

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "j"] \arrow[d, "\alpha"] & Z \arrow[r, "q"] \arrow[d, "\bar{\alpha}"] & X \arrow[r] \arrow[d, equal] & 0 & {[z]} \\
0 \arrow[r] & E \arrow[r, "\iota"] & PO \arrow[r, "\pi"] & X \arrow[r] & 0 & {[\alpha z]}
\end{tikzcd}$$

- Let $\alpha' = \iota\alpha$, which clearly admits an extension $\bar{\alpha} : Z \to PO$.
- If there were a compact $\beta : Z \to PO$ extending α, then

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "j"] \arrow[d, "\alpha"] & Z \arrow[r, "q"] \arrow[d, "\beta"] & X \arrow[r] \arrow[d, "\gamma"] & 0 & {[z]} \\
0 \arrow[r] & E \arrow[r, "\iota"] & PO \arrow[r, "\pi"] & X \arrow[r] & 0 & {[\alpha z]}
\end{tikzcd}$$

for some $\gamma : X \to X$ necessarily compact.

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "j"] \arrow[d, "\alpha"] & Z \arrow[r, "q"] \arrow[d, "\beta"] & X \arrow[r] \arrow[d, "\gamma"] & 0 & {[\mathrm{z}]} \\
0 \arrow[r] & E \arrow[r, "\iota"] & PO \arrow[r, "\pi"] & X \arrow[r] & 0 & {[\alpha \mathrm{z}]}
\end{tikzcd}$$

- This diagram can be expanded as

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "j"] \arrow[d, "\alpha"] & Z \arrow[r] \arrow[d] \arrow[ddd, bend right, "\beta"'] & X \arrow[r, "q"] \arrow[d, equal] & 0 & {[\mathrm{z}]} \\
0 \arrow[r] & E \arrow[r] \arrow[d, equal] & PO \arrow[r] \arrow[d, dotted] & X \arrow[r] \arrow[d, equal] & 0 & {[\alpha \mathrm{z}]} \\
0 \arrow[r] & E \arrow[r] \arrow[d, equal] & PB \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "\gamma"] & 0 & {[\alpha \mathrm{z} \gamma]} \\
0 \arrow[r] & E \arrow[r, "\iota"] & PO \arrow[r, "\pi"] & X \arrow[r] & 0 & {[\alpha \mathrm{z}]}
\end{tikzcd}$$

which states that the pull-back of $[\alpha \mathrm{z}]$ with γ is equivalent to $[\alpha \mathrm{z}]$ itself.

- The finite ascent property of $\mathfrak{K}$ implies $[\alpha \mathrm{z}]$ is trivial.

Thank you very much
for your attention!