– Automatic extension properties for free quasi-Banach lattices –

Alberto Salguero Alarcón

Universidad Complutense de Madrid / Instituto de Ciencias Matemáticas

joint work with P. Tradacete and N. Trejo-Arroyo

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Contents

The topic

Example

Can every operator from L_1 to L_p for some $0 < p < 1$ be extended to a lattice homomorphism $\mathrm{FBL}[L_1] \to L_p$?

- Choose E a vector space and let $\mathcal{F}[E]$ be some free lattice generated by E.
- Then $\mathcal{F}[E]$ is characterized by a universal property of extension (preserving the norm, if applicable) of maps $E \to X$, where X belongs to certain class of target spaces $\mathcal{X}$.

Then we can ask:

1. Does $\mathcal{F}[E]$ also extends maps $E \to X$ where X belongs to a bigger class $\mathcal{X}'$?
2. If it does, what does this imply for $\mathcal{F}[E]$?

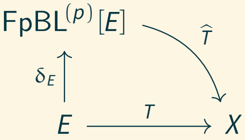
A brief reminder

- A quasi-Banach lattice X is *p-convex* $(0 < p \leqslant +\infty)$ if there exists $C > 0$ such that

$$\left\| \left(\sum_{k=1}^{n} |x_k|^p \right)^{\frac{1}{p}} \right\| \leqslant C \left(\sum_{k=1}^{n} \|x_k\|^p \right)^{\frac{1}{p}}$$

  for every $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$.
  The infimum of such C is denoted $M^{(p)}(X)$.
- A quasi-Banach space E is *p-natural* for some $0 < p \leqslant 1$ if it can be isometrically embedded in a p-convex p-Banach space X with $M^{(p)}[X] = 1$.
- A p-natural quasi-Banach space is automatically p-Banach.
- Every p-Banach space is r-Banach for any $0 < r < p$.

Fix $p \in (0, 1]$ and let E be a p-natural quasi-Banach space.
The *free p-convex p-Banach lattice* generated by E is a p-convex p-Banach lattice $\mathrm{FpBL}^{(p)}[E]$ with $M^{(p)}(\mathrm{FpBL}^{(p)}[E]) = 1$ such that it enjoys the following universal property:



where

- X is a p-convex p-Banach space.
- $\|\widehat{T}\| \leqslant M^{(p)}[X]\|T\|$.

Actually, it suffices to ask norm-preserving extensions for operators $T : E \to L_p$:

The main problem

Question A

Given $0 < q < p$, is there a constant $C_{p,q} > 0$ such that every operator $T : E \to L_q$ can be extended to a lattice homomorphism $\widehat{T} : \mathrm{FpBL}^{(p)}[E] \to L_q$ with $\|\widehat{T}\| \leqslant C_{p,q}\|T\|$?

Compatibility

- Let E be a quasi-Banach space which is p-natural for some p.
- Then E is also r-natural for all $r < p$.
- Consider

$$p_E = \sup\{p \in (0, 1] : E \text{ is } p\text{-natural}\}.$$

(Observe that E does *not* need to be p_E-natural).

- Then, $\mathrm{FrBL}^{(r)}[E]$ exists for every $0 < r < p_E$...
- ...and if E is p_E-natural, then it also exists for $r = p_E$.

Two models

- $M^{(r)}[E]$ is a non-decreasing function on r, and it is defined either on $(0, p_E)$ or on $(0, p_E]$.

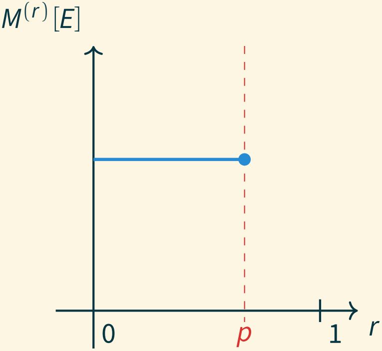


Figure 1: $M^r[L_p]$.

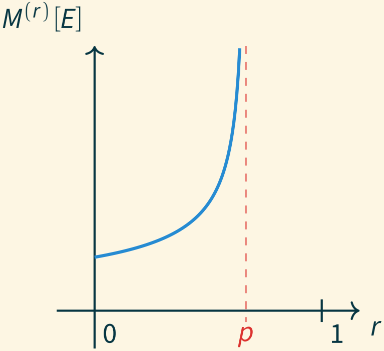


Figure 2: $M^r[L_{p,\infty}]$.

Compatibility conditions between $\mathrm{FrBL}^{(r)}[E]$

Question B

Is there any relation between $\mathrm{FrBL}^{(r)}[E]$ and $\mathrm{FpBL}^{(p)}[E]$ for any $0 < r < p < p_E$?
More precisely,

- Are they isomorphic?
- Is $\mathrm{FrBL}^{(r)}[E]$ p-convex?

If E is p_E-natural, then the above problem also makes sense for $p = p_E$.

Only one problem

Actually, Questions A and B are (essentially) equivalent.

Proposition

Let E be a natural quasi-Banach space and let $r < p < p_E$. The following are equivalent:

1. $\mathrm{FrBL}^{(r)}[E]$ is lattice-isomorphic to $\mathrm{FpBL}^{(p)}[E]$.
2. $\mathrm{FrBL}^{(r)}[E]$ is p-convex.
3. There exists $C > 0$ such that every operator $T : E \to L_r$ can be extended to a lattice homomorphism $\widehat{T} : \mathrm{FpBL}^{(p)}[E] \to L_r$ with $\|\widehat{T}\| \leqslant C\|T\|$.

If E is p_E-natural, then one can also take $p = p_E$ above.

Contents

The main theorem

Theorem

Let E be a natural quasi-Banach space. Given $0 < p < p_E$, there is a constant $\Lambda_p(E)$ such that, for every $0 < r < p$, the spaces $\mathrm{FrBL}^{(r)}[E]$ and $\mathrm{FpBL}^{(p)}[E]$ are lattice $\Lambda_p(E)$-isomorphic.

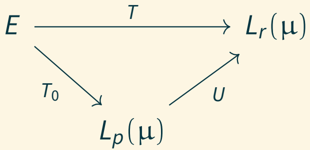
In other words, for a fixed $p < p_E$, there is a constant $C_p > 0$ such that every operator $T : E \to L_r$ can be extended to a lattice homomorphism $\widehat{T} : \mathrm{FpBL}^{(p)}[E] \to L_r$ with $\|\widehat{T}\| \leqslant C_p \|T\|$.

- Observe that the previous assertion is automatic for $r \geqslant p$.

Ingredients of the proof

A version of Maurey's factorization theorem

Let E be a natural quasi-Banach space. Given $0 < r < p < p_E \leqslant 1$, there exists a constant $C_{r,p}[E]$ such that every operator $T : E \to L_r(\mu)$ admits a factorization of the form



where

- $\|T_0\| \leqslant C_{r,p}[E]\|T\|$.
- U is a norm-one lattice homomorphism.

Ingredients of the proof

> ***An asymptotic study***
> Let E be a natural quasi-Banach space and $0 < r < p < p_E \leqslant 1$. Then $\lim_{r\to 0^+} C_{r,p}[E] < \infty$.

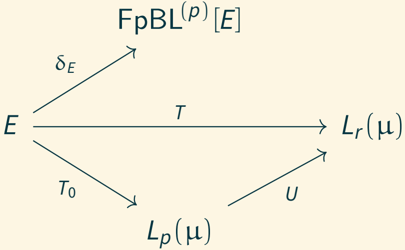
Actually, one can take, for any $q \in (p, p_E)$,

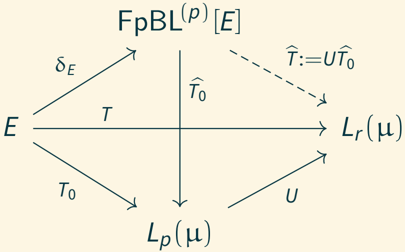
$$C_{r,p}[E] \leqslant T_q[E] \cdot \frac{A_{p,q}}{A_{r,q}},$$

where

- $T_q[E]$ is the constant of type q of E.
- $A_{p,q}$ is the norm in L_p of a q-stable random variable.

A bit of diagram chasing



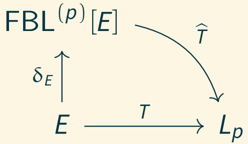


Contents

Extrapolation theorems

These ideas were already studied in the Banach setting:

- For $p \in [1, +\infty]$, let $\mathrm{FBL}^{(p)}[E]$ denote the *free p-convex Banach lattice* over a Banach space E.



Theorem (Oikberg, Taylor, Tradacete, Troitsky)

Suppose $\mathrm{FBL}^{(p)}[E]$ is q-convex for some $1 \leqslant p < q$. Then, $\mathrm{FBL}^{(r)}[E]$ is q-convex for all $1 \leqslant r \leqslant +\infty$.

Extrapolation theorems

In other words, the following statements are equivalent:

1. every operator $E \to L_p$ can be extended to a lattice homomorphism $\mathrm{FBL}^{[q]}[E] \to L_p$ (in a norm-preserving way).
2. every operator $E \to X$, where X is **any Banach** lattice can be extended to a lattice homomorphism $\mathrm{FBL}^{[q]}[E] \to X$ (in a norm-preserving way and up to $M^{(p)}[X]$).

We add:

3. every operator $E \to X$, where X is **any natural quasi-Banach** lattice can be extended to a lattice homomorphism $\mathrm{FBL}^{[q]}[E] \to X$ (in a quasi-norm-preserving way and up to $M^{(p)}[X]$).

Sharpness

Proposition

There are operators $L_1 \to L_{1,\infty}$ which cannot be extended to $\mathrm{FBL}[L_1]$.

The simplest example seems to be the Hilbert transform.

- Then, $\mathrm{FBL}[L_1]$ is not isomorphic to $\mathrm{FpBL}^{(p)}[L_1]$ for any $p < 1$.
- However, $\mathrm{FpBL}^{(p)}[L_1]$ and $\mathrm{FrBL}^{(r)}[L_1]$ are isomorphic for every $r, p < 1$.

Question 1

Given $0 < p < 1$, is there an operator $L_p \to L_{p,\infty}$ which cannot be extended to $\mathrm{FpBL}^{(p)}[L_p]$?

Question 2

Given $p > 1$, is there an operator $L_p \to L_{p,\infty}$ which cannot be extended to $\mathrm{FBL}^{(p)}[L_p]$?

Why not consider the class of natural quasi-Banach spaces *as a whole*?

Definition (Kalton)

A quasi-Banach lattice X is *L-convex* if it is p-convex for some $p \in (0, +\infty]$.

Since the function $r \mapsto M^{(r)}(X)$ is non-increasing and bounded from below, it make sense to define the *constant of L-convexity* for a quasi-Banach space X as

$$M^{(L)}(X) = \inf_{r \to 0^+} M^{(r)}(X).$$

More free quasi-Banach lattices...?

Definition

Given a natural quasi-Banach space E, the *free L-convex quasi-Banach lattice generated by E* is a pair $(\mathrm{FQBL}^{(\mathrm{L})}[E], \delta)$ where:

- $\delta : E \to \mathrm{FQBL}^{(\mathrm{L})}[E]$ is an into isometry,
- $\mathrm{FQBL}^{(\mathrm{L})}[E]$ is an L-convex quasi-Banach lattice $\mathrm{FQBL}^{(\mathrm{L})}[E]$ with $M(\mathrm{FQBL}^{(\mathrm{L})}[E]) = 1$,

such that every operator $T : E \to X$, where X is an L-convex quasi-Banach lattice, admits a unique extension through δ to a lattice homomorphism

$$\widehat{T} : \mathrm{FQBL}^{(\mathrm{L})}[E] \to X$$

with $\|\widehat{T}\| \leqslant M^{(L)}(X)\|T\|$.

Actually, no!

Theorem

For every natural quasi-Banach space E, $\mathrm{FQBL}^{(\mathsf{L})}[E]$ exists, and it is lattice isomorphic to $\mathrm{FpBL}^{(p)}[E]$ for any $p < p_E$.

Note: the constants of the isomorphisms $\mathrm{FQBL}^{(\mathsf{L})}[E] \simeq \mathrm{FpBL}^{(p)}[E]$ may blow up as $p \to p_E$.

Upper p-estimates

Question 3

Is there a *"free quasi-Banach lattice with upper p-estimates"*?

The main obstacle:

- There are very nasty quasi-Banach lattices, but *every* quasi-Banach lattice satisfies some kind of upper p-estimate!

More precisely:

Theorem (Kalton)

For a quasi-Banach lattice X, we have:

$$X \text{ is } p\text{-Banach} \;\Rightarrow\; X \text{ has upper } p\text{-estimate} \;\Rightarrow$$
$$\Rightarrow\; X \text{ is } q\text{-Banach for } \tfrac{1}{q} = \tfrac{1}{p} + 1.$$

Moreover, the implications are optimal, and none of the arrows can be reversed.

A big Thank you to the organizers

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Thank you all very much

for your attention!